

Polynomials

Resultants

Resultant of Two Polynomials

$$Q(X) = AX^2 + 2BX + C$$

$$R(X) = DX^2 + 2EX + F$$

$$\mathbf{R}(Q, R) = f(A, B, C, D, E, F)$$

$\mathbf{R} = 0 \iff Q$ and R have a common root

Calculating the Resultant

$$Q(X) = AX^2 + 2BX + C = 0$$

$$R(X) = DX^2 + 2EX + F = 0$$

$$aQ + bR = 0$$

$$DQ - AR = 0$$

$$\begin{aligned} & D(AX^2 + 2BX + C) \\ & - A(DX^2 + 2EX + F) = 0 \end{aligned}$$

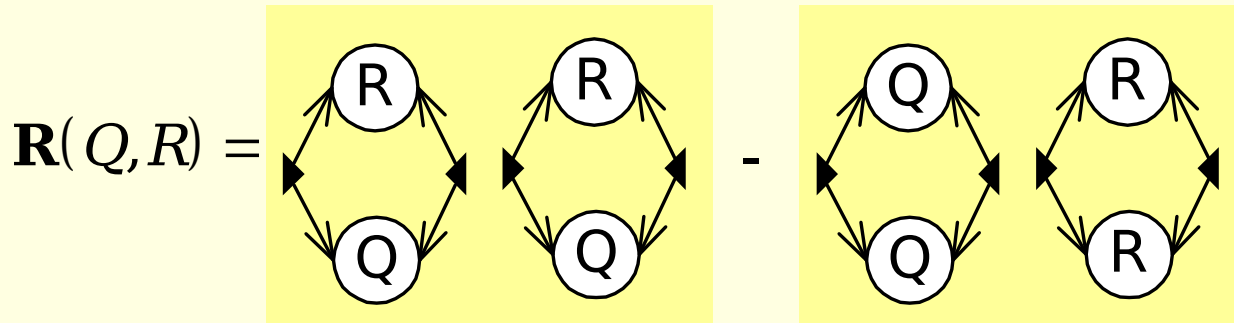
$$2(BD - AE)X + (CD - AF) = 0$$

Resultant of Q and R

$$Q(x, w) = Ax^2 + 2Bxw + Cw^2$$

$$R(x, w) = Dx^2 + 2Exw + Fw^2$$

$$\mathbf{R}(Q, R) = +A^2F^2 - 4ABEF + 4ACE^2 - 2ACDF \\ + 4B^2DF - 4BCED + C^2D^2$$



Two Forms of Resultant

$$\mathbf{R}(Q, R) = \begin{array}{|c|c|} \hline \begin{array}{c} \text{R} \\ \text{Q} \end{array} \\ \hline \end{array} - \begin{array}{|c|c|} \hline \begin{array}{c} \text{Q} \\ \text{R} \end{array} \\ \hline \end{array}$$

$$\mathbf{R}(Q, R) = \begin{array}{|c|c|} \hline \begin{array}{c} 2 \\ \text{R} \\ \text{Q} \end{array} \\ \hline \end{array} - \begin{array}{|c|c|} \hline \begin{array}{c} 2 \\ \text{Q} \\ \text{R} \end{array} \\ \hline \end{array}$$

Identities:

$$\begin{array}{|c|} \hline \begin{array}{c} 2 \\ \text{Q} \\ \text{R} \end{array} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \begin{array}{c} \text{Q} \\ \text{Q} \end{array} \\ \hline \end{array} \quad \begin{array}{|c|c|} \hline \begin{array}{c} \text{R} \\ \text{R} \end{array} \\ \hline \end{array}$$

$$\begin{array}{|c|c|} \hline \begin{array}{c} 2 \\ \text{R} \\ \text{Q} \end{array} \\ \hline \end{array} = \begin{array}{|c|c|} \hline \begin{array}{c} \text{R} \\ \text{Q} \end{array} \\ \hline \end{array} - \begin{array}{|c|c|} \hline \begin{array}{c} \text{Q} \\ \text{R} \end{array} \\ \hline \end{array}$$

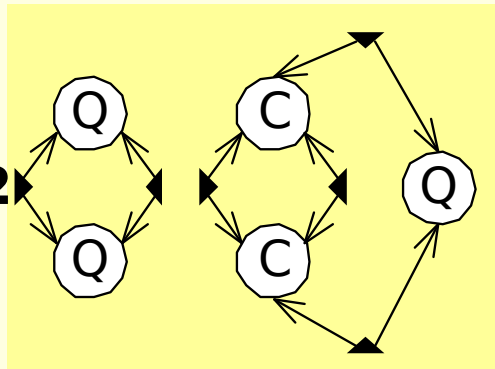
Resultant of Q and C

$$C(x, w) = Ax^3 + 3Bx^2w + 3Cxw^2 + Dw^3$$

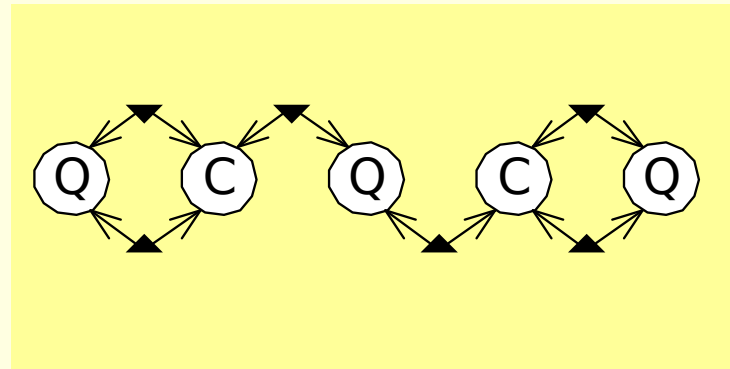
$$Q(x, w) = Ex^2 + 2Fwx + Gw^2$$

$$\begin{aligned} R(Q, C) = & -A^2(G^3) + 6AB(FG^2) - 6AC(2F^2G - EG^2) - 2AD(-4F^3 + 3EFG) \\ & - 9B^2(EG^2) + 18BC(EFG) - 6BD(2EF^2 - E^2G) \\ & - 9C^2(E^2G) + 6CD(E^2F) \\ & - D^2(E^3) \end{aligned}$$

$$R(Q, C) = -2$$



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Forms of Q,C Resultant

$$R(Q,C) = -2 \left[\begin{array}{c} \text{Diagram 1} \end{array} \right] - \left[\begin{array}{c} \text{Diagram 2} \end{array} \right]$$

Diagram 1: A graph with 6 nodes (3 Q, 3 C) arranged in two vertical columns of three. Each column has a cycle of three nodes. Horizontal edges connect corresponding nodes in the two columns. Vertical edges connect nodes within each column.

Diagram 2: A graph with 6 nodes (3 Q, 3 C) arranged in two horizontal rows of three. Each row has a cycle of three nodes. Horizontal edges connect corresponding nodes in the two rows. Vertical edges connect nodes within each row.

$$R(Q,C) = -3/2 \left[\begin{array}{c} \text{Diagram 3} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 4} \end{array} \right]$$

Diagram 3: A graph with 6 nodes (3 Q, 3 C) arranged in two vertical columns of three. Each column has a cycle of three nodes. Horizontal edges connect corresponding nodes in the two columns. Vertical edges connect nodes within each column.

Diagram 4: A graph with 6 nodes (3 Q, 3 C) arranged in two horizontal rows of three. Each row has a cycle of three nodes. Horizontal edges connect corresponding nodes in the two rows. Vertical edges connect nodes within each row.

Identit

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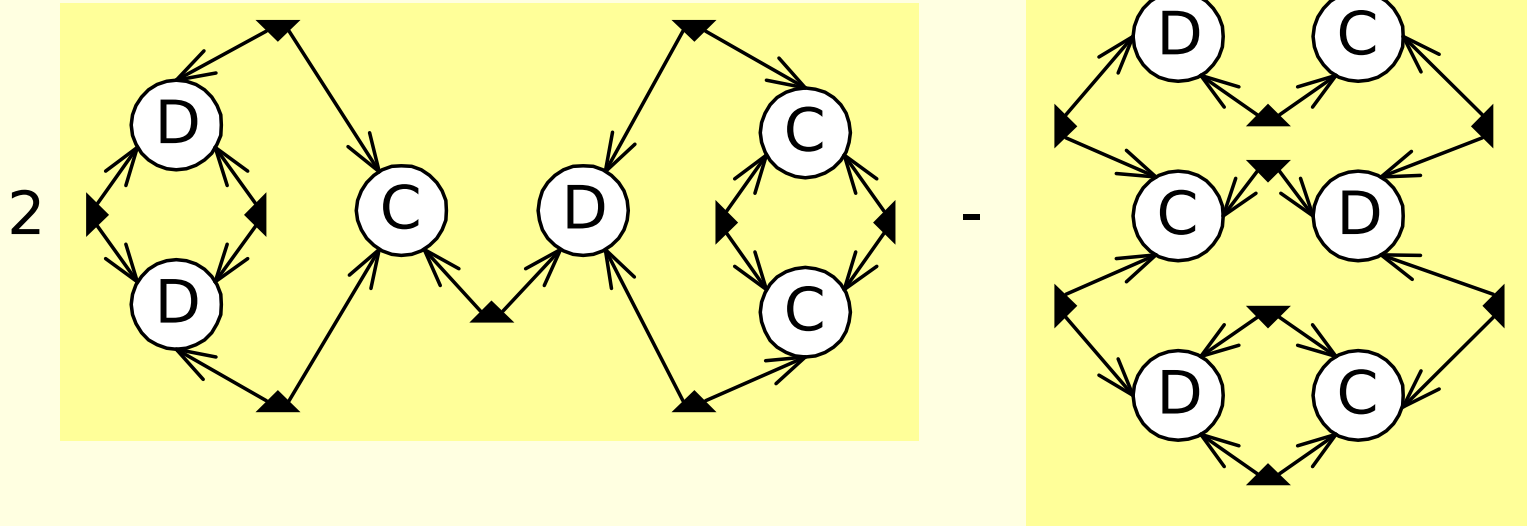
$$\left[\begin{array}{c} \text{Diagram 5} \end{array} \right] = - \left[\begin{array}{c} \text{Diagram 6} \end{array} \right] + \left[\begin{array}{c} \text{Diagram 7} \end{array} \right]$$

Diagram 5: A graph with 6 nodes (3 Q, 3 C) arranged in two horizontal rows of three. Each row has a cycle of three nodes. Horizontal edges connect corresponding nodes in the two rows. Vertical edges connect nodes within each row.

Diagram 6: A graph with 6 nodes (3 Q, 3 C) arranged in two horizontal rows of three. Each row has a cycle of three nodes. Horizontal edges connect corresponding nodes in the two rows. Vertical edges connect nodes within each row.

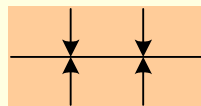
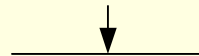
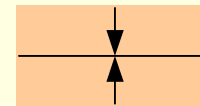
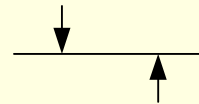
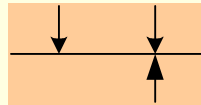
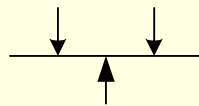
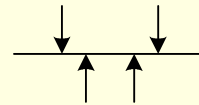
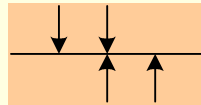
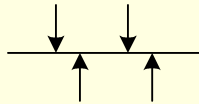
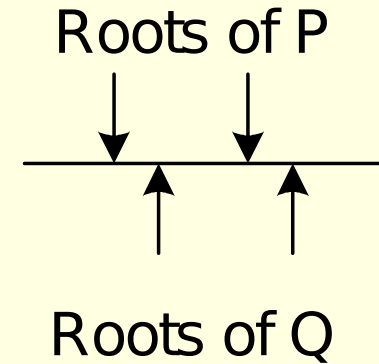
Diagram 7: A graph with 6 nodes (3 Q, 3 C) arranged in two horizontal rows of three. Each row has a cycle of three nodes. Horizontal edges connect corresponding nodes in the two rows. Vertical edges connect nodes within each row.

Resultant of two Cubics ?



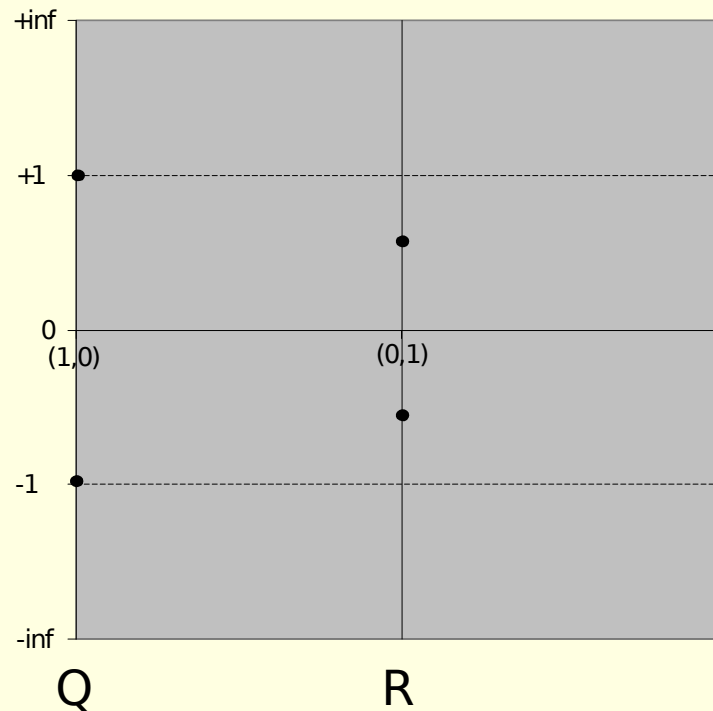
What's Really Going on With Resultants?

Possible Relationships between P and Q



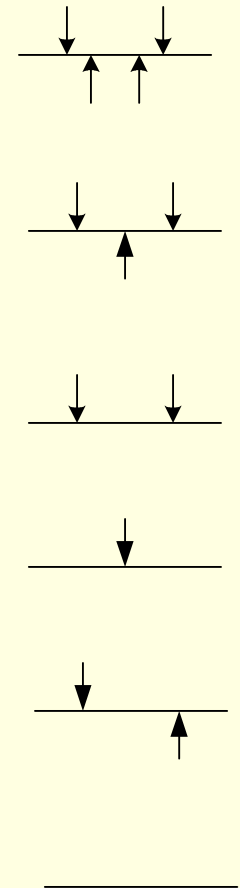
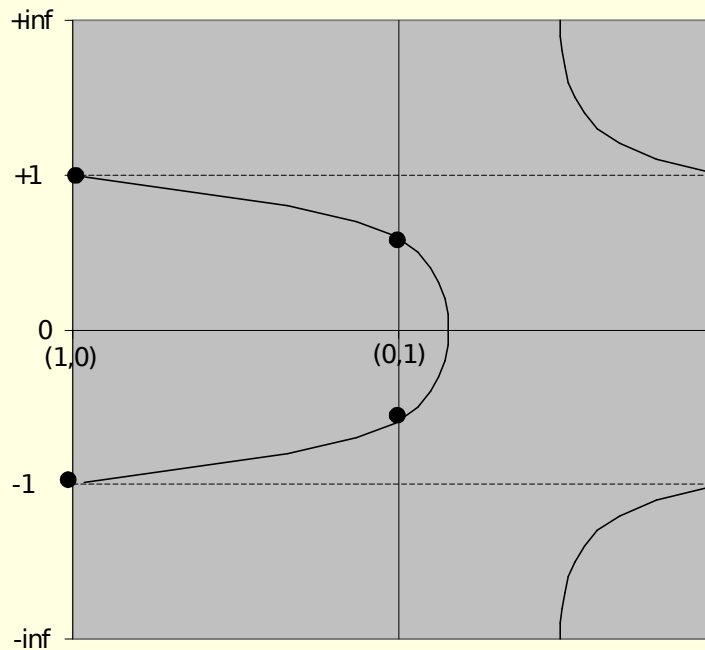
Linear Combos of Q and R

$$aQ + bR = 0$$



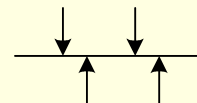
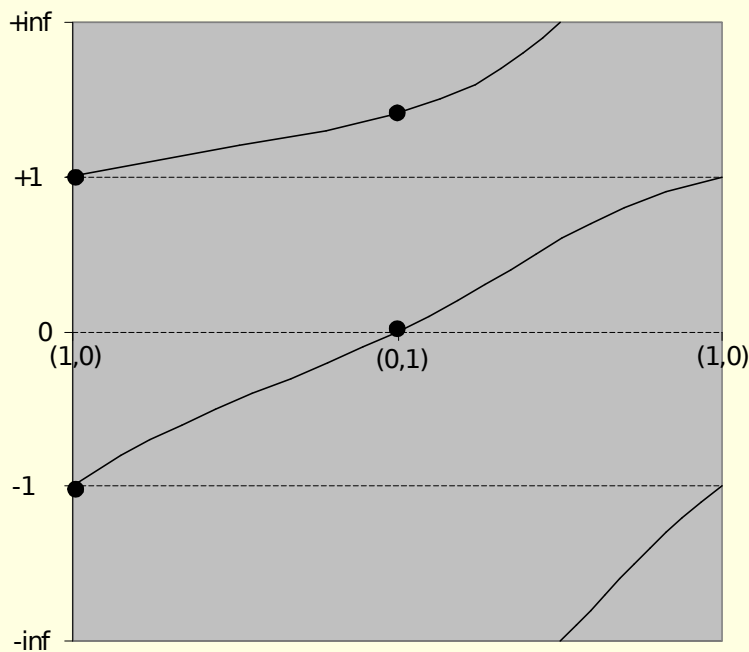
Q,R Have Enclosed Roots

$$aQ + bR = 0$$



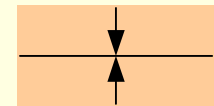
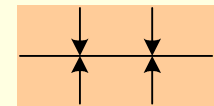
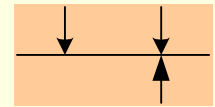
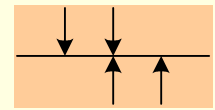
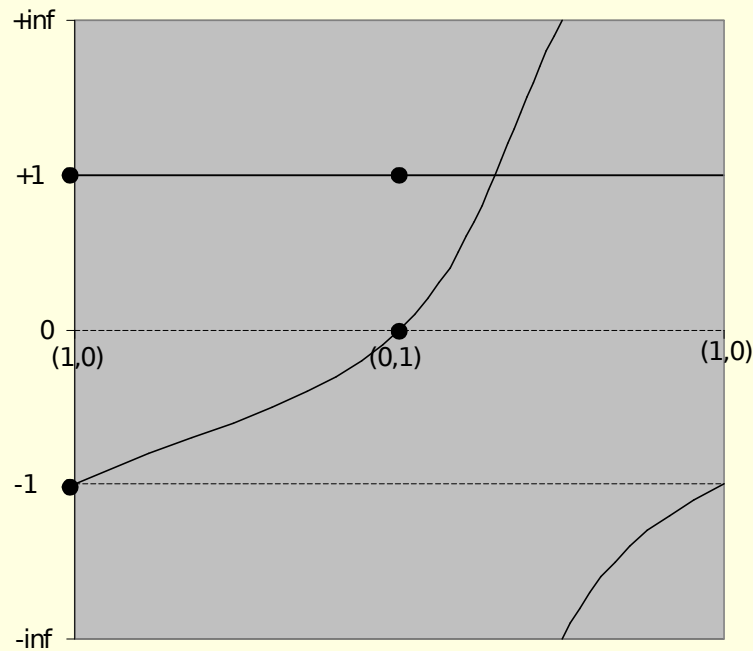
Q,R Have Interleaved Roots

$$aQ + bR = 0$$

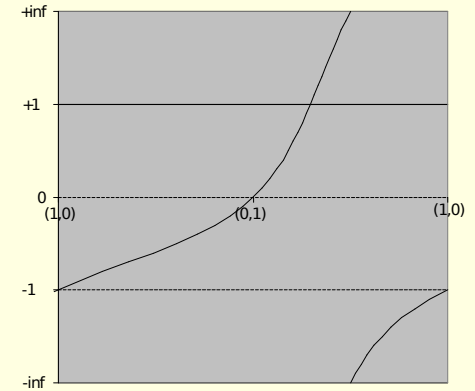
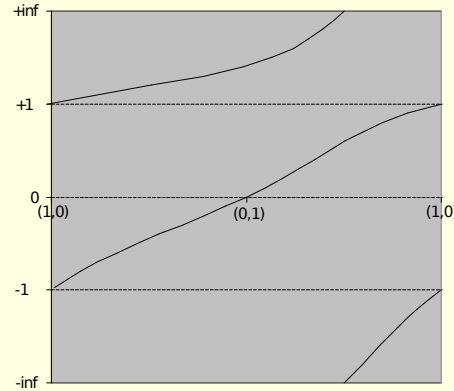
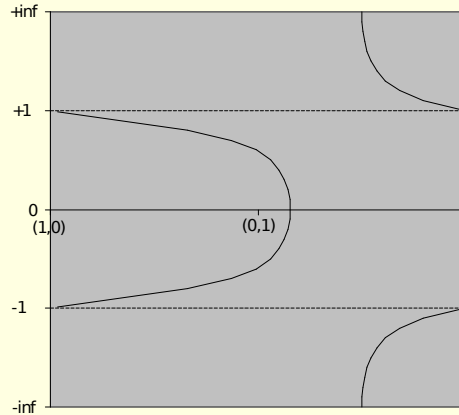


Q,R Have Common Root

$$aQ + bR = 0$$



Three Possible Evolutions of Roots of $aQ + bR$

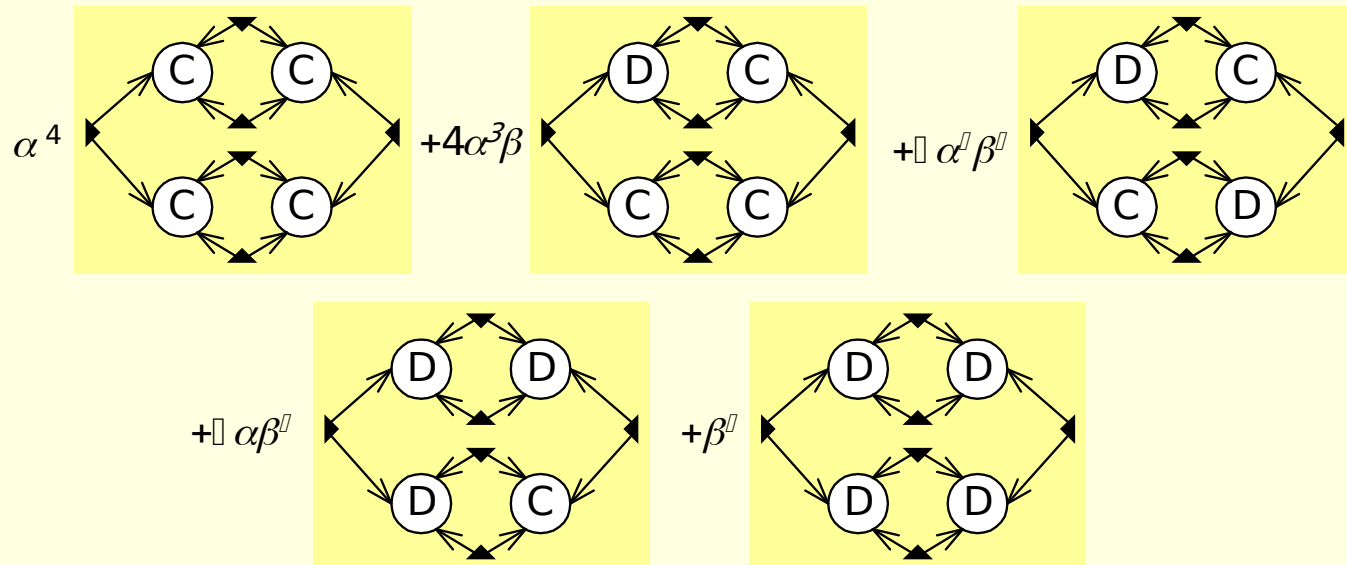


$$\det(aQ + bR) = \alpha^2 \begin{array}{c} \text{Q} \\ \updownarrow \\ \text{Q} \end{array} + 2\alpha\beta \begin{array}{c} \text{R} \\ \updownarrow \\ \text{Q} \end{array} + \beta^2 \begin{array}{c} \text{R} \\ \updownarrow \\ \text{R} \end{array}$$

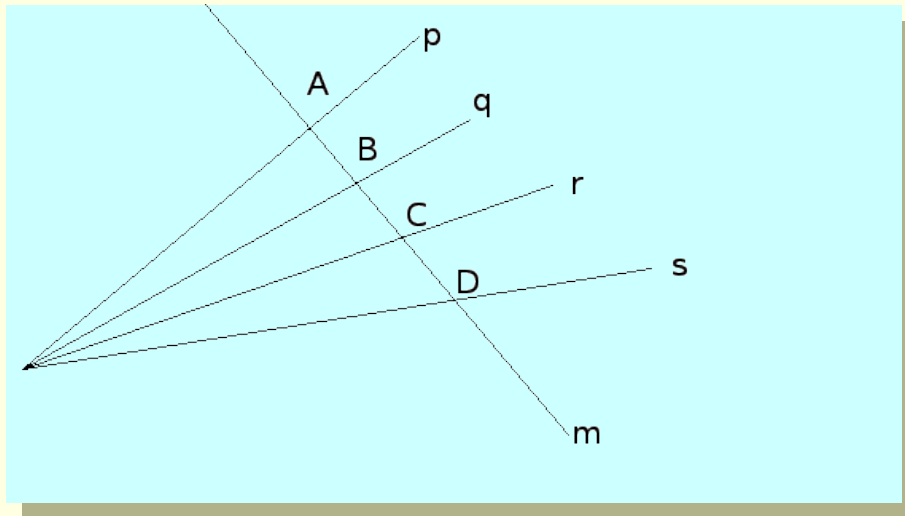
$$\mathbf{R}(Q, R) = \begin{array}{c} \text{R} \\ \updownarrow \\ \text{Q} \end{array} \begin{array}{c} \text{R} \\ \updownarrow \\ \text{Q} \end{array} - \begin{array}{c} \text{Q} \\ \updownarrow \\ \text{Q} \end{array} \begin{array}{c} \text{R} \\ \updownarrow \\ \text{R} \end{array}$$

Possible Evolutions of Roots of Two Cubics

$$\det(a\mathbf{C} + b\mathbf{D}) =$$



The Cross Ratio



$$\chi = \frac{|AB|/|BD|}{|AC|/|CD|}$$

$$c = \frac{(p' q \times m)(r' s \times m)}{(q' s \times m)(p' r \times m)}$$

Generalized Cross Ratio of Two Quadratic Polynomials

$$\chi = \frac{\begin{array}{|c|c|} \hline \begin{array}{c} \text{R} \\ \text{Q} \end{array} \\ \hline \end{array}}{\begin{array}{|c|c|} \hline \begin{array}{c} \text{Q} \\ \text{Q} \end{array} \\ \hline \end{array}}$$

The diagram illustrates the generalized cross ratio χ as a ratio of two configurations of points. The numerator consists of two identical cycles, each containing a top node labeled 'R' and a bottom node labeled 'Q'. The denominator consists of two identical cycles, each containing a top node labeled 'Q' and a bottom node labeled 'R'. In all cycles, the nodes are connected by four directed arrows forming a diamond shape: two arrows point from the top node to the bottom node, and two arrows point from the bottom node to the top node.